

Non-markovian global persistence in phase-ordering kinetics

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Abstract. The persistence probability $P_g(t)$ of the global order-parameter of a simple ferromagnet undergoing phase-ordering kinetics after a quench from a fully disordered state to below the critical temperature, $T < T_c$, is analysed. It is argued that the persistence probability decays algebraically with time in the entire low-temperature phase. For Markov processes, the associated global persistence exponent $\theta_g = (2\lambda_C - d)/(2z)$ is related to the autocorrelation exponent λ_C . This relationship is confirmed for phase-ordering in the exactly solved 1D Ising model and the d -dimensional spherical model. For the 2D Glauber-Ising model, the temperature-independent estimate $\theta_g = 0.063(2)$ indicates that the dynamics of the global order-parameter is described by a non-Markovian process.

PACS numbers: 05.70.Ln, 05.50.+q, 64.60.Ht, 75.40.Gb

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Persistence properties of a stochastic process have been studied in a variety of dynamical systems, ranging from magnetic systems [1, 2, 3, 4, 5, 6, 7, 8, 9] to soap bubbles [10], from reaction-diffusion systems [11] to fluctuating interfaces [12], turbulent liquid crystals [13] and non-equilibrium surface growth [14]. The persistence probability $P(t)$ is defined in a very general way as the probability that a given stochastic variable (observable) retains a characteristic feature over the time t . One commonly studied problem involves the question whether the stochastic variable, which can be a local or a global quantity, maintains its sign up to the time t .

Over the years, the persistence properties of the fluctuations around the mean value of the local order parameter $\phi(t)$ have attracted much interest. Problems of this kind were initially formulated for the probability $P_\ell(t)$ that the local order-parameter $\phi(t, \mathbf{r})$ has not yet changed its sign up to time t . For large enough times, one expects a power-law decay $P_\ell(t) \sim t^{-\theta_\ell}$ if the system is at an equilibrium critical point, and where θ_ℓ is the *local* persistence exponent. This has been studied for simple diffusion [15, 16] and for one-dimensional exactly solvable models quenched to $T = 0$ [1, 2, 4, 5, 6], for example in the q -states Potts model one has the remarkable result $\theta_\ell = -\frac{1}{8} + [\frac{2}{\pi} \arccos(\frac{2-q}{\sqrt{2q}})]^2/2$ [5]. Values of θ_ℓ for higher-dimensional models have been extracted from numerical simulations, see [3] for data on the 2D Ising model.

It has turned out to be very useful [7] to study the persistence properties of the *global* order-parameter

$$\hat{\phi}_0(t) = \frac{1}{\sqrt{|\Omega|}} \int_{\Omega} d\mathbf{r} \phi(t, \mathbf{r}) \quad (1)$$

averaged over spatial domains Ω with large volumes $1 \ll |\Omega| \ll |V|$, where $|V|$ is the total volume of the system. The global persistence probability $P_g(t)$ is defined as the probability that $\hat{\phi}_0(t)$ has not changed sign up to time t . For large times, one expects a power-law decay $P_g(t) \sim t^{-\theta_g}$ which defines the *global* persistence exponent θ_g . This scenario was shown to apply in particular to non-equilibrium critical dynamics ($T = T_c$) [7]. Here, an application of the central limit theorem shows that $\hat{\phi}_0(t)$ should be described by a Gaussian stochastic process and if that process is furthermore Markovian, one has the scaling relation $\theta_g = \lambda_C - d + 1 - \eta/2$ [7]. The definition of the exponents η and λ_C will be recalled below. For critical systems, this relation is satisfied in a few exactly solvable systems such as the 1D Glauber-Ising model quenched to $T = 0$ or the spherical model in $2 < d < 4$ dimensions (for all of which the dynamical exponent happens to be $z = 2$). However, the scaling relation does break down in more generic systems, where $z \neq 2$ in all known examples. One concludes [7] that the process describing the long-time collective dynamics of $\hat{\phi}_0(t)$ and which arises from renormalised field-theory, is in general non-Markovian, see [17] for a review. This has been confirmed by a large variety of analytical and simulational studies, either for critical ferromagnets (including effects of disorder) [18] or else for genuine non-equilibrium systems at an absorbing phase transition [19]. More recent developments consider the cross-over in the global persistence as a function of the initial value of the magnetisation [20] or the behaviour in the vicinity of surfaces [21, 22] or on sub-manifolds [23].

Aging phenomena, which are observed in a broad variety of systems with slow relaxation dynamics [24, 25, 26, 27, 28], are paradigmatic examples of systems that are far from stationarity, and of great potential for applications. Initially found in glassy systems, they also occur in simple ferromagnets, which are often considerably easier to analyse. This kind of non-equilibrium collective phenomena may be conveniently realized by quenching a system from a fully disordered initial state to a temperature $T \leq T_c$, where T_c is the critical temperature. The specific aspects of aging, namely (i) slow, non-exponential relaxation, (ii) breaking of time-translation-invariance and (iii) dynamical scaling, are conveniently studied through the behaviour of two-time quantities such as the two-time correlator $C(t, s; \mathbf{r}) = \langle \phi(t, \mathbf{r}) \phi(s, \mathbf{0}) \rangle$ of the order parameter. For simple ferromagnets, one finds a dynamical scaling regime when both times are sufficiently large and if $t - s \gg t_{\text{micro}}$, where t_{micro} is a microscopic reference time. Then

$$C(t, s; \mathbf{r}) = s^{-b} f_C(t/s, |\mathbf{r}|^z/(t - s)) \quad (2)$$

where $b = 0$ for $T < T_c$ and $b = (d - 2 + \eta)/z$ for $T = T_c$, where η describes the decay of the critical equilibrium correlations and z is the dynamical exponent. For $y = t/s$ large, one has $f_C(y, \mathbf{0}) \sim y^{-\lambda_C/z}$ where λ_C is the autocorrelation exponent [29, 30], whose value depends on whether $T < T_c$ or $T = T_c$.

In this work, we wish to return to an analysis of the properties of the global persistence for simple ferromagnets which undergo phase-ordering kinetics after a quench from a fully disordered initial state, with a vanishing average initial global magnetisation $\langle \hat{\phi}_0(0) \rangle = 0$, to $T < T_c$ with $T_c > 0$.§ It has been known since a long time that $z = 2$ [31, 32] and current theories for the growth laws in phase-ordering kinetics usually start from a local differential equation $\partial_t \phi = -\delta \mathcal{F} / \delta \phi$ where \mathcal{F} is a local Landau-Ginzburg functional. Since the temperature $T < T_c$ is thought to be irrelevant, one sets $T = 0$ and analyses the long-time behaviour of the solutions of the above equation, after having averaged over the totally disordered initial state [31, 32, 24]. Although this set-up is perfectly local in time, the behaviour of the global persistence probability is non-trivial in general. We find:

- (i) The global order-parameter $\hat{\phi}_0(t)$ is described by a Gaussian stochastic process and its global persistence probability $P_g(t) \sim t^{-\theta_g}$ decays algebraically.
- (ii) If the process for $\hat{\phi}_0(t)$ is also Markovian, then $\theta_g = \theta_g^{\text{mark}}$, where [8, 9]

$$\theta_g^{\text{mark}} z = \lambda_C - d/2 \geq 0 \quad (3)$$

The bound follows from the Yeung-Rao-Desai inequality [33].

- (iii) Eq. (3) is satisfied for several exactly solvable systems such as the 1D Glauber-Ising model at $T = 0$ [7] and the spherical model at $T < T_c$ and for $d > 2$. These systems are explicitly Markovian.

§ For an ordered initial state, however, one expects $P_g(t) \rightarrow P_{g,\infty}(T)$ exponentially fast in t and without dynamical scaling. For $T < T_c$, there is no analogue to the cross-over observed in [20] for critical quenches.

- (iv) While the results stated so far come from essentially standard procedures, we also performed simulations of the 2D Glauber-Ising model quenched to $T < T_c \approx 2.27$ and find

$$\theta_g = \begin{cases} 0.062(2) & ; \text{ for } T = 1.0 \\ 0.065(2) & ; \text{ for } T = 1.5 \end{cases} \quad (4)$$

These results do not agree with eq. (3). Hence, the process $\hat{\phi}_{\mathbf{0}}(t)$ is not Markovian in general. This example of a presumably non-integrable system also illustrates that the simple value $z = 2$ of the dynamic exponent, as it holds true in phase-ordering kinetics [31, 32], does not imply the Markov property.

Generalising the lines of thought developed in [7] to quenches below T_c [8, 9], consider the formation of ordered domains. Shortly after the quench, they should be rather small, with a linear size ξ_0 at most of the order of the lattice constant. For later times, the firmly established dynamical scaling [24] (for a proof of dynamical scaling in 2D phase-ordering, see [34]) leads to a *single* relevant time-dependent length-scale $L(t) \sim t^{1/z}$ and the dynamics of the system is described by the movement of the domain walls of independent domains of linear sizes larger than $L(t)$, where the volume $|V| \gg L(t)^d$ of the system must be large enough. For any given time t , the first two moments of the global order-parameter are finite and read $\langle \hat{\phi}_{\mathbf{0}}(t) \rangle = 0$ and $\langle \hat{\phi}_{\mathbf{0}}^2(t) \rangle \sim t^{d/z}$. Since one averages over the configuration space of disordered initial states \parallel , the central limit theorem is applicable, hence the stochastic process describing the time-evolution of $\hat{\phi}_{\mathbf{0}}(t)$ should be Gaussian, also for $T < T_c$ [8, 9]. Again because of dynamical scaling, the two-time global correlator is $\langle \hat{\phi}_{\mathbf{0}}(t) \hat{\phi}_{\mathbf{0}}(s) \rangle = s^{d/z} \hat{f}_C(t/s)$ with $t > s$ and the asymptotic behaviour $\hat{f}_C(y) \sim y^{(d-\lambda_C)/z}$ as $y \rightarrow \infty$. Therefore the normalised global autocorrelator ($t > s$)

$$N(t, s) := \frac{\langle \hat{\phi}_{\mathbf{0}}(t) \hat{\phi}_{\mathbf{0}}(s) \rangle}{\sqrt{\langle \hat{\phi}_{\mathbf{0}}^2(t) \rangle \langle \hat{\phi}_{\mathbf{0}}^2(s) \rangle}} = \hat{f}_N(t/s) \quad (5)$$

has the asymptotic form $\hat{f}_N(y) \sim y^{-(\lambda_C - d/2)/z}$ as $y \gg 1$. Changing temporal variables according to $T = \ln t$, the process describing $\hat{\phi}_{\mathbf{0}}$ becomes stationary. According to Doob's lemma [36], in this setting the Markov property is equivalent to having the *exact* correlator $N(t, s) = \bar{N}(T, S) = \exp(-\mu|T - S|)$, with $\mu = (2\lambda_C - d)/(2z)$. Then Slepian's formula [37] shows that the global persistence probability indeed decays with time according to $P_g(t) \sim t^{-\theta_g^{\text{mark}}}$ such that eq. (3) holds true.

Not surprisingly, eq. (3) can be confirmed in some exactly solvable models the dynamics of which are explicitly Markovian. As a first example, we consider the 1D Glauber-Ising model quenched to $T = 0$ from a fully disordered state. Since $T_c = 0$ in 1D, the global persistence in this model was already considered as an example for a critical quench [7], but we shall see that the same result may also be interpreted in a low-temperature setting. Indeed, from the well-known results $\lambda_C = 1$ and $z = 2$

\parallel If that average is not taken, simple scaling may be replaced by multi-scaling, see e.g. [35]

[38], eq. (3) gives the Markovian prediction $\theta_g^{\text{mark}} = \frac{1}{4}$. The exact calculation gives the expected power-law decay and $\theta_g = \frac{1}{4}$ [7], as it should be.

As a second example, we consider the spherical model in $d > 2$ dimensions quenched to $T < T_c$. The spherical model may be formulated in terms of real-valued spin variables $S_{\mathbf{r}}$ subject to the constraint $\sum_{\mathbf{r}} S_{\mathbf{r}}^2 = \mathcal{N}$, with the total number \mathcal{N} of lattice sites. The Hamiltonian describes ferromagnetic nearest-neighbour interactions and reads $\mathcal{H} = -\sum_{(\mathbf{r}, \mathbf{r}')} S_{\mathbf{r}} S_{\mathbf{r}'}$. The non-conserved dynamics is given by a Langevin equation, with a totally disordered initial state. Following the usual lines of its exact solution [39], the two-time spin-spin correlator at momentum \mathbf{q} reads $\hat{C}_{\mathbf{q}}(t, s) = \hat{C}_{\mathbf{q}}(s) e^{-\omega(\mathbf{q})(t-s)} \sqrt{g(s)/g(t)}$ where

$$\begin{aligned} \hat{C}_{\mathbf{q}}(t) &= \frac{\exp(-2\omega(\mathbf{q})t)}{g(t)} \left[\hat{C}_{\mathbf{q}}(0) + 2T \int_0^t d\tau e^{2\omega(\mathbf{q})\tau} g(\tau) \right] \\ g(t) &= f(t) + 2T \int_0^t d\tau f(t-\tau) g(\tau) \end{aligned} \quad (6)$$

with $f(t) = (e^{-4t} I_0(4t))^d$ and I_0 is a modified Bessel function, while $\omega(\mathbf{q}) = 2 \sum_{i=1}^d (1 - \cos q_i) \simeq q^2$. Then the normalised global correlator becomes

$$N(t, s) = \frac{\hat{C}_{\mathbf{0}}(t, s)}{\sqrt{\hat{C}_{\mathbf{0}}(t) \hat{C}_{\mathbf{0}}(s)}} = \sqrt{\frac{\hat{C}_{\mathbf{0}}(s) g(s)}{\hat{C}_{\mathbf{0}}(t) g(t)}}. \quad (7)$$

For a fully disordered initial state $\hat{C}_{\mathbf{0}}(0) = 1$.

Indeed, for zero temperature $T = 0$, we readily have $N(t, s) = 1$, in agreement with the explicit Markov property in this model. In order to show that this result remains true for all $0 \leq T < T_c$, we write first $\hat{C}_{\mathbf{0}}(t) g(t) = 1 + 2T \int_0^t d\tau g(\tau)$ and then solve the Volterra integral equation (6) [39, 40], which leads for large enough times to $\int_0^t d\tau g(\tau) = g_0 + g_1 t^{1-d/2}$, where the explicitly known values of the constants $g_{0,1}$ will not be needed. Hence

$$N(t, s) = 1 + \mathcal{O}(s^{1-d/2}) \quad (8)$$

indeed follows an exact power law in t/s , up to corrections to dynamical scaling, and Slepian's formula [37] gives $\theta_g = 0$. Since the autocorrelation exponent $\lambda_C = d/2$ [39], this agrees with the Markovian prediction (3), as expected.¶

Finally, we study the global persistence in the 2D Glauber-Ising model, quenched to $T < T_c$ from a fully random initial state with zero magnetisation. In Fig. 1 we show the global persistence probability $P_g(t)$, and also compare results for different values of T . The global persistence probability is simply the probability that the global magnetisation has not crossed zero until time t . Finite-size effects were monitored carefully by simulating systems of various system sizes. The data in Fig. 1 have been obtained for a system with 400×400 sites and result from an average over 80,000 different

¶ A similar conclusion holds true for the *long-range* spherical model, with Hamiltonian $\mathcal{H} = -\sum_{\mathbf{r} \neq \mathbf{r}'} J(\mathbf{r} - \mathbf{r}') S_{\mathbf{r}} S_{\mathbf{r}'}$ and $J(\mathbf{r}) \sim |\mathbf{r}|^{-d-\sigma}$ with $0 < \sigma < 2$, quenched to $T < T_c$. For non-conserved dynamics, one has $z = \sigma$, $\lambda_C = d/2$ [41], and a straightforward extension of the above calculation shows that $N(t, s) = 1$, up to finite-time corrections, and that eq. (3) is satisfied.

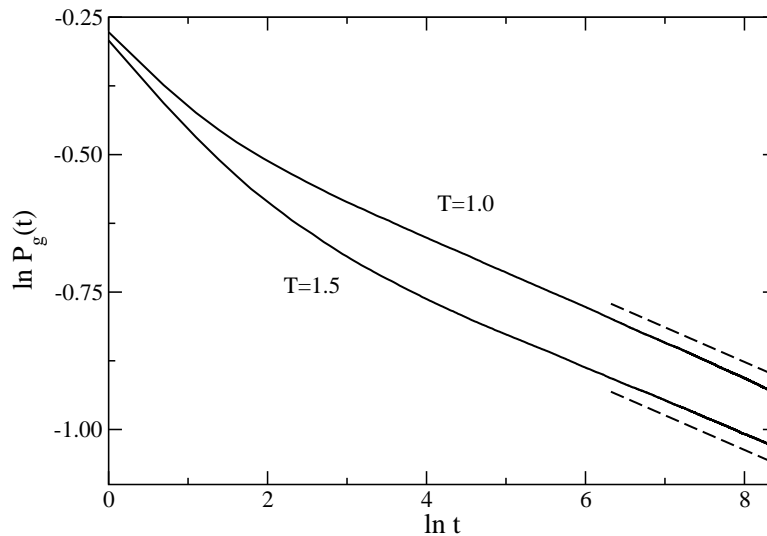


Figure 1. Global persistence probability for the 2D Glauber-Ising model, quenched either to $T = 1.0$ or to $T = 1.5$. After an initial regime at short times a power-law regime is observed, with an exponent $\theta_g \approx 0.063$ (dashed lines). Data were averaged over 80,000 different runs with different initial states and different realizations of the noise.

runs where we averaged over different initial conditions [35] and different realizations of the noise. Clearly, for sufficiently large times the decay of the global persistence is described by a power-law. The values of θ_g extracted are listed in eq. (4). However, since $\lambda_C = 1.25(2)$ [26, 28], the estimates (4) are quite distinct from the Markovian prediction $\theta_g^{\text{mark}} = 0.125(2)$. Therefore, the effective long-time dynamics of the global order-parameter should *not* be described by a Markov process. This is our main result.

The data in Fig. 1 seem to indicate the existence of an initial short-time regime, followed by the power-law decay. In fact, as we start from a disordered initial state, the dynamical correlation length $L(t)$ increases as a function of time. One then expects that the power-law behaviour characteristic of the $T = 0$ fixed point should dominate the persistence probability only once $L(t) \gg \xi$ where ξ is the equilibrium correlation length which sufficiently close to T_c should scale as $\xi \sim (T_c - T)^{-\nu}$. As long as $L_0 \ll L(t) \ll \xi$, where L_0 is a microscopic length scale, the leading behaviour of $P_g(t)$ should be the same as at the critical point. In order to verify this, we monitored the initial time behaviour as a function of temperature and found that the initial regime can be described by a power-law decay with a temperature-dependent *effective* exponent. The value of this exponent (as well as the cross-over time between the short-time regime and the late-time regime, see Fig. 1) increases with increasing T . We find the values 0.18 for $T = 1.8$ and 0.20 for $T = 2.0$, already close to the value $\theta_g = 0.237(3)$ [7, 42, 20] of the global persistence exponent at the critical point. If $L(t)$ is of the order of L_0 , the evolution will depend on all sorts of non-universal, microscopic details, and no general, model-independent statement can be made.

In Fig. 2 we show the normalised correlator $N(t, s)$ for different waiting times s at the temperature $T = 1.5$. We first verify in Fig. 2a that $N(t, s)$ is indeed only a

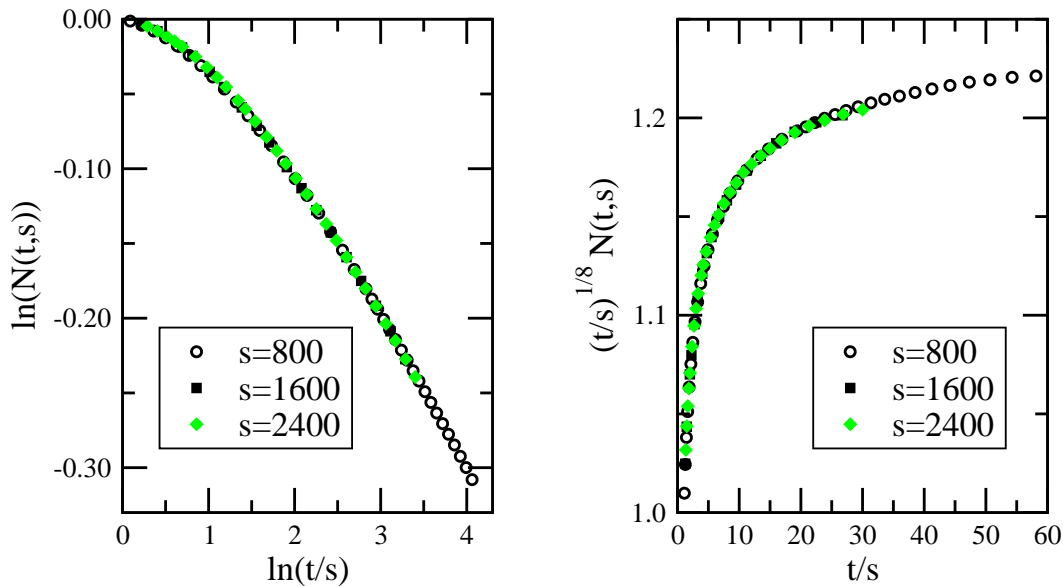


Figure 2. (Colour online) Normalised correlator $N(t,s)$ for the 2D Glauber-Ising model, quenched to $T = 1.5$. (a) Data obtained for different waiting times fall on a common master curve when plotted as a function of t/s . The effective exponent at large times is ≈ 0.115 , already close to the expected value 0.125. (b) Plotting $(t/s)^{0.125} N(t,s)$ over against t/s shows that even for the largest accessible values of t/s , the asymptotic power-law regime is not yet completely reached. The error bars, which result from averaging over at least 5,000 independent runs, are much smaller than the symbol sizes.

function of t/s , see (5). As we are exploring the time-dependence of the correlator up to $t/s = 60$, we have to simulate rather large systems with 800×800 spins in order to avoid finite-size effects. While in the accessed time-window the value of the effective exponent of the asymptotic decay ≈ 0.115 is already close to the expected value 0.125, we also find that this correlator does not follow a simple power law which gives additional evidence against $\hat{\phi}_0$ being described by a Markov process.

Our result (4) for θ_g is consistent with data on the block persistence at $T = 0$ [8, 9], where $\theta_g \approx 0.09$ in the 2D Ising model and $\theta_g \approx 0.06$ in the 2D time-dependent Ginzburg-Landau equation was found, in agreement with the expectation that T should be an irrelevant parameter.

We observe that while for quenches to $T = T_c$, all known systems give $\theta_g \geq \theta_g^{\text{mark}}$, see e.g. [7, 19, 17, 20, 21, 22, 28, 42], the presently available evidence suggests that the opposite inequality $\theta_g \leq \theta_g^{\text{mark}}$ might hold true for quenches into the low-temperature phase $T < T_c$ (here, we exclude cases with $T_c = 0$ from the discussion).

Summarising, we have studied the extension of a scaling relation [7, 8, 9] for the global persistence exponent θ_g , valid for Markovian systems, from critical systems to phase-ordering kinetics. We have found evidence, through the breaking of the scaling relation (3) and the non-trivial form of the normalised global two-time correlator $N(t,s)$, that the time-dependent global order-parameter $\hat{\phi}_0$ should in general *not* be described by a Markovian stochastic process, unless the underlying model is integrable. The example

of the 2D Ising model quenched to $T < T_c$ illustrates explicitly that non-markovian dynamics and a simple value $z = 2$ are independent properties of non-equilibrium critical systems. It is tempting to speculate that the generically non-markovian dynamics might be related to the fact that in phase-ordering kinetics the width of the boundaries between the ordered domains increases with time as $w(t) \sim t^\kappa$, with $\kappa = \frac{1}{4}$ in the 2D Ising model [43], and which might lead to some ‘memory effect’ in the causal interactions of the coarse-grained order-parameter across the interfaces. Testing this further would require precise simulational data for a quench right to $T = 0$ (when the roughening would be absent) and by comparing with our results, obtained for $T > 0$. Further studies of the non-critical behaviour of the global persistence $P_g(t)$ would be very welcome.

Acknowledgments

We thank S.N. Majumdar and C. Sire for useful correspondence and A. Rosso for a discussion. MH thanks the *groupe de travail maths/physique* in Nancy for stimulating questions. This work was supported in part by the US National Science Foundation through DMR-0904999.

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